Ergodic Theory and Measured Group Theory
Lecture 11

A **growth** of a finitely-generated group Γ is defined as follows: fix a finite generating set S and for each $n \in \mathbb{N}$, let $r_S(n) := |B_n|$, where B_n is the ball of radius n centered at the identity in $\mathrm{Cay}(\Gamma, S)$. This is asymptotically well-defined because for a different generating set S', $r_S \leq O(r_{S'}(n))$ and $r_{S'}(n) \in O(r_S(n))$.

Examples.
- For $\mathbb{Z}^d$, $r(n) := n^d$.
- For $\mathbb{F}_d$, $r(n) := 2d \cdot (2d-1)^{n-1}$, so exponential.

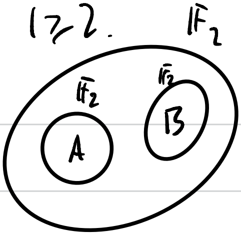
Prop. Groups of subexponential growth are amenable.

Proof. Exercise.

But there are amenable groups of exponential growth, e.g. the lamplighter group.

Nonamenable groups. The canonical example is $\mathbb{F}_2$. Why? Because it's paradoxical, i.e. there are two disjoint copies

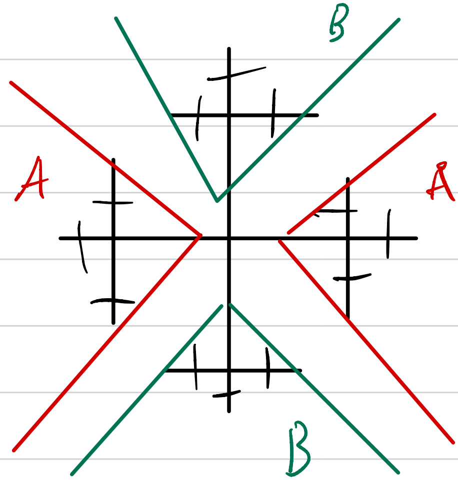
$1 \geq 2.$



of $\mathbb{F}_2$ inside $\mathbb{F}_2$ and each of these copies is a piecewise translate of $\mathbb{F}_2$. Let $A := [a] \sqcup [a^{-1}]$, $B := [b] \sqcup [b^{-1}]$.

$a^{-1} \cdot [a] = [a^{-1}]^c$ so

$$\begin{array}{ccccc} A := & [a] & \sqcup & [a^{-1}] \\ \downarrow & \downarrow a^{-1} & & \downarrow 1 \\ \mathbb{F}_2 := & [a^{-1}]^c & \sqcup & [a^{-1}] \end{array}$$



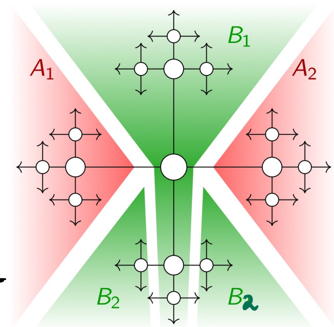
Similarly,

$$\begin{array}{cccc} B := & [b] & \sqcup & [b^{-1}] \\ & \downarrow b^{-1} & & \\ \mathbb{F}_2 = & [b^{-1}]^c & \sqcup & [b^{-1}]. \end{array}$$

Thus, if there was an invariant prob. measure λ on $\mathbb{F}_2$, then $\lambda(A) = \lambda(\mathbb{F}_2) = 1$ and $\lambda(B) = 1$, so $1 \geq 1+1$.

Def. For a group Γ, subsets $A, B \subseteq \Gamma$ are called *finitely-equidecomposible* (or just *equidecomposible*) if $\exists$ finite partitions $A = A_1 \sqcup \dots \sqcup A_n$ and $B = B_1 \sqcup B_2 \sqcup \dots \sqcup B_n$ s.t. each B_i is a translate of A_i, i.e. $B_i = \gamma_i A_i$ for some $\gamma_i \in \Gamma$. A decomposition $\Gamma = A \sqcup B$ is called *paradoxical* if A and B are both equidecomposible with Γ.

Technically, we didn't show that $\mathbb{F}_2$ admits a paradoxical decomposition because $A \cup B = \mathbb{F}_2 \setminus \{1\}$, but one can take $A = [a] \sqcup [a^{-1}]$, $B := B_1 \sqcup B_2$, where $B_1 := [b] \cup b^{-\infty}$, $B_2 = [b^{-1}] \setminus b^{-\infty}$, $b^{-\infty} := \{b^{-n} : n \in \mathbb{N}\}$. Then easy to see that B is still equidecomposible with $\mathbb{F}_2$: $b^{-1} \cdot B_1 \sqcup 1 \cdot B_2 = \mathbb{F}_2$. Hence $\mathbb{F}_2 = A \cup B$ is a paradoxical decomposition.



It's obvious (from the invariant prob. measure definition) that if a group admits a paradoxical decomposition then it's nonamenable.

Theorem (Tarski). A group is nonamenable if and only if it admits a paradoxical decomposition.

Recall that amenability is closed under subgroups, so if $\mathbb{F}_2 \hookrightarrow \Gamma$, then Γ is nonamenable.

Von Neumann – Day Problem (1957 paper of Day). Is it true that a group Γ is nonamenable $\iff$ $\mathbb{F}_2 \hookrightarrow \Gamma$?

Tits alternative: True for linear groups, like $GL_n(\mathbb{Z})$: either such a group is virtually solvable (hence amenable) or $\mathbb{F}_2 \hookrightarrow$ in it.

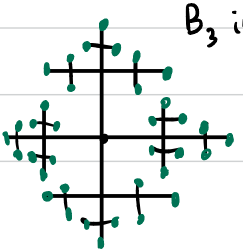
Ol'shanski (1980). No! He constructed a Tarski monster that is nonamenable but $\mathbb{F}_2 \not\hookrightarrow$.

Monod (2013). Easier group (called Frankenstein group) and descriptive set theoretic proof through actions.

Going back to ergodic theorems: we said that what makes the proof of ergodic theorems work is the Følner property of the sequence (F_n) of finite subsets of the group Γ along which the averages are taken. So our proofs fail for nonamenable groups. Turns out not just the proofs...

Example (Tao 2015). The pointwise ergodic theorem (for L^1) along the balls fails for some pmp action of $\mathbb{F}_2$. In fact, $\exists \geq 0$ L^1-function s.t. averages along the spheres are unbounded (hence also along balls because a sphere is $\geq \frac{1}{2}$ ball).

B_3 in $\mathbb{F}_2$



To still get an ergodic theorem, we artificially make the boundary of balls small by assigning weights to elements of $\mathbb{F}_2$ so that each sphere get total weight 1. For example, the assignment trough the symmetric nonbacktracking random walk: $m_s(w_0 w_1 \dots w_n) := \frac{1}{4 \cdot 3^n}$ for any reduced word $w_0 \dots w_n \in \mathbb{F}_2$.

Grigorchuk 1988, Nevo 1994. Let m_s be the symmetric assignment of weights on $\mathbb{F}_2$. For any pmp action a of $\mathbb{F}_2$ on a st. prob. space (X, μ), for any $f \in L^1(X, \mu)$,

$$\lim_{n \to \infty} \frac{m_s\text{-weighted average of } f \text{ over } B_n \cdot x}{m_s(B_n)} = \mathbb{E}(f \mid \mathcal{B}_a).$$

$$\|$$

$$\lim_{n \to \infty} \frac{\sum_{\gamma \in B_n} f(\gamma \cdot x) \cdot m_s(\gamma)}{n+1} = \lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} (\text{average of } f \text{ over } S_i)$$

where S_i is the sphere of radius i.

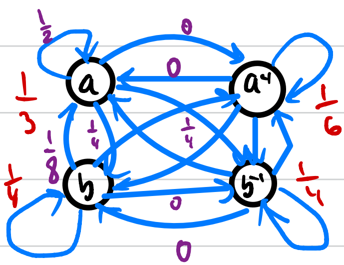
What about other kinds of assignments of weights on $\mathbb{F}_2$ that still give each sphere a total weight 1. Such a class of assignments is given by Markov Chains.

Def (in elementary terms). A Markov chain with a state space S (for us a ctbl set) is an assignment of weights on $S^{<\mathbb{N}}$:= the set of all finite words in S, given by

$$m(w_0 w_1 \dots w_n) := \pi(w_0)\cdot P(w_0, w_1)\cdot P(w_1, w_2)\cdots P(w_{n-1}, w_n),$$

where π is a probability vector, i.e. $\pi : S \to [0,1]$ s.t. (thinking of π as a row vector and $\mathbb{1}$ as a column vector constant 1), $\pi \cdot \mathbb{1} = 1$, i.e. $(\pi_1, \pi_2, \dots, \pi_7)\cdot\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = 1$.

And $P : S \times S \to [0,1]$ s.t. each row adds up to 1.



Example. $S := \{a, a^{-1}, b, b^{-1}\}$.

$\pi := \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$ — initial distribution
(a, a^{-1}, b, b^{-1})

$$P := \begin{array}{c} \\ a \\ a^{-1} \\ b \\ b^{-1} \end{array}\begin{array}{cccc} a & a^{-1} & b & b^{-1} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ & \vdots & & \\ & & & \\ & & & \end{array}$$ — transition matrix